LIMIT MODELS IN METRIC ABSTRACT ELEMENTARY CLASSES: THE CATEGORICAL CASE

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Abstract. We study versions of limit models adapted to the context of *metric abstract elementary classes*. Under categoricity and superstability-like assumptions, we generalize some theorems from [GrVaVi]. We prove criteria for existence and uniqueness of limit models in the metric context.

1. Preliminaries - Why limit models and towers?

The Model Theory of metric structures can be generalized in an effective way to the Abstract Elementary Class (AEC) context by blending some of the constructions typical of AECs with ideas and paradigms from First Order Continuous Model Theory as understood by [BeBeHeUs] and in other senses benefitting from the enormous wealth of Stability Theory in Abstract Elementary Classes. Other authors (Hirvonen [Hi] in her thesis with Hyttinen, and also independently Usvyatsov and Shelah) have provided other frameworks for model theory of metric structures outside continuous first order.

Hirvonen and Hyttinen have developed a solid framework for categoricity transfer of metric AEC and for the study of $\aleph_0$-stable classes of metric structures (a good analysis of primary models, basic items in the definition itself, etc.).

Our focus here is the beginning of an analysis of "superstability" in metric AEC. Of course this goal is long-winded, but we provide first steps in that direction in this paper. In particular, building mainly on ideas from the discrete AEC setting coming from [GrVaVi], and related more distantly to Shelah's ideas in [Sh600], we approach here the connection

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between two facets of (protean) superstability: limit models (existence and first steps towards uniqueness).

The main constructions in our paper are versions of towers adapted to the metric context (s-towers and metric s-towers). Specifically, reduced towers have been used extensively by Grossberg, Shelah, VanDieren, the first author of this paper and Jarden in their work on AEC before. Here we adapt them to the metric setting and use them to prove various lemmas useful to an approach of uniqueness of limit models in metric AEC (in a forthcoming paper). Towers can be regarded as a strong generalization of the concept of Galois type: a Galois type is (an equivalence class) of triples (M, N, a) where $M \prec N$ and $a \in N \setminus M$ – towers "refine" the way the element a is connected to M inside N and provides a very robust situation where a is replaced by a long sequence $(a_i)_i$, and the models M and N themselves are "sliced through". Extension properties of triples, and ultimately, independence and "forking" calculus-like properties of the triples may be lifted in a robust way to towers. This has been explored by the authors mentioned above in the usual AEC setting - we begin the exploration here of the *metric* version.

Towers play a key role in the proof of uniqueness of limit models given in [GrVaVi]. In this paper, we prove -under μ^+-d-categoricity- *uniqueness -up to isomorphism- of limit models of density character* μ (i.e., if M_1 is a (μ, θ_1)-limit model over M and M_2 is a (μ, θ_2)-limit model over M and $dc(M_1) = dc(M_2)$, then $M_1 \approx_M M_2$) under suitable superstability-like assumptions. If $cf(\theta_1) = cf(\theta_2)$, then by a standard *back and forth* argument we are done. So, if $cf(\theta_1) \neq cf(\theta_2)$, as in [GrVaVi], the key idea is to build a (μ, θ)-limit model over M M_θ which is also a (μ, ω)-limit model over M for any ordinal $\theta < \mu^+$, so

$$\begin{aligned} M_1 &\approx_M M_{\theta_1} \text{ (because they are } (\mu, \theta_1)\text{-limits over } M) \\ &\approx_M M_{\theta_2} \text{ (because they are } (\mu, \omega)\text{-limits over } M) \\ &\approx_M M_2 \text{ (because they are } (\mu, \theta_2)\text{-limits over } M) \end{aligned}$$

In order to build a model such that (μ, θ)-limit model over M M_θ which is also a (μ, ω)-limit model over M for any ordinal $\theta < \mu^+$, as in [GrVaVi], we define the notion of *smooth tower*, which corresponds to an adaptation of the notion of *tower* given in [GrVaVi] but in our metric setting. The key idea is to extend (via $\mathcal{K}$-embeddings) a given tower of length of cofinality θ to a special kind of tower (*reduced towers*) which is continuous and to a kind of tower (*relatively full tower*) which satisfies a kind of relative saturation. Iterating this argument ω many times, the idea is to prove that the directed limit of such directed system is a *reduced* (and therefore a continuous) tower where the completion of its union is a

(μ, θ)-limit model over M (which is consequence of the *full-relativeness* of the extensions given in the directed system). To be a (μ, ω)-limit model over M is assured defining in a suitable way the notion of extension of towers. Although this argument has the same general outline as the proof done in [GrVaVi], we point out that many details in our proof here are quite different: e.g., our notion of *reduced towers* involves a Lipschitz-like function which determines a notion of closeness of towers instead of intersections as in [GrVaVi] and we have to deal with completion of union of increasing chains of towers in the metric sense instead of just its union, which makes more complicated some of the arguments if we compare them with the proofs given in [GrVaVi].

In [GrVaVi] the authors prove the uniqueness of limit models under superstability-like assumptions for AEC. Here we study the behavior of s-towers under superstability-like assumptions for the metric setting.

2. Basic facts on Metric AECs

We consider a natural adaptation of the notion of *Abstract Elementary Class* (see [Gr] and [Ba]), but work in a context of Continuous Logic that generalizes the "First Order Continuous" setting of [BeBeHeUs] by removing the assumption of uniform continuity[1]. We base our definitions on [Hi, GrVa].

Definition 2.1. The *density character* of a topological space is the smallest cardinality of a dense subset of the space. If X is a topological space, we denote its density character by $dc(X)$. If A is a subset of a topological space X, we define $dc(A) := dc(\overline{A})$.

Definition 2.2. Let $\mathcal{K}$ be a class of L-structures (in the context of Continuous Logic) and $\prec_{\mathcal{K}}$ be a binary relation defined in $\mathcal{K}$. We say that $(\mathcal{K}, \prec_{\mathcal{K}})$ is a *Metric Abstract Elementary Class* (shortly *MAEC*) if:

(1) $\mathcal{K}$ and $\prec_{\mathcal{K}}$ are closed under isomorphism.
(2) $\prec_{\mathcal{K}}$ is a partial order in $\mathcal{K}$.
(3) If $M \prec_{\mathcal{K}} N$ then $M \subseteq N$.
(4) (*Completion of Union of Chains*) If $(M_i : i < \lambda)$ is a $\prec_{\mathcal{K}}$-increasing chain then
  (a) the function symbols in L can be uniquely interpreted on the completion of $\bigcup_{i<\lambda} M_i$ in such a way that $\overline{\bigcup_{i<\lambda} M_i} \in \mathcal{K}$
  (b) for each $j < \lambda$, $M_j \prec_{\mathcal{K}} \overline{\bigcup_{i<\lambda} M_i}$
  (c) if each $M_i \in \mathcal{K} \in N$, then $\overline{\bigcup_{i<\lambda} M_i} \prec_{\mathcal{K}} N$.

[1]Uniform continuity guarantees logical compactness in their formalization, but we drop compactness in AEC-like settings.

(5) (*Coherence*) if $M_1 \subseteq M_2 \prec_{\mathcal{K}} M_3$ and $M_1 \prec_{\mathcal{K}} M_3$, then $M_1 \prec_{\mathcal{K}} M_2$.
(6) (DLS) There exists a cardinality $LS^d(K)$ (which is called the *metric Löwenheim-Skolem number*) such that if $M \in \mathcal{K}$ and $A \subseteq M$, then there exists $N \in \mathcal{K}$ such that $dc(N) \leq dc(A) + LS^d(K)$ and $A \subseteq N \prec_{\mathcal{K}} M$.

Definition 2.3. We call a function $f : M \to N$ a $\mathcal{K}$-*embedding* if

(1) For every k-ary function symbol F of L, we have $f(F^M(a_1 \cdots a_k)) = F^N(f(a_1) \cdots f(a_k))$.
(2) For every constant symbol c of L, $f(c^M) = c^N$.
(3) For every m-ary relation symbol R of L, for every $\bar{a} \in M^m$, $d(\bar{a}, R^M) = d(f(\bar{a}), R^N)$.
(4) $f[M] \prec_{\mathcal{K}} N$.

Definition 2.4 (Amalgamation Property, AP)**.** Let $\mathcal{K}$ be an MAEC. We say that $\mathcal{K}$ satisfies *Amalgamation Property* (for short *AP*) if and only if for every $M, M_1, M_2 \in \mathcal{K}$, if $g_i : M \to M_i$ is a $\mathcal{K}$-embedding $(i \in \{1,2\})$ then there exist $N \in \mathcal{N}$ and $\mathcal{K}$-embeddings $f_i : M_i \to N$ $(i \in \{1,2\})$ such that $f_1 \circ g_1 = f_2 \circ g_2$.

$$\begin{tikzcd} M_1 \arrow[r, dotted, "f_1"] & N \\ M \arrow[u, "g_1"] \arrow[r, "g_2"'] & M_2 \arrow[u, dotted, "f_2"'] \end{tikzcd}$$

Definition 2.5 (Joint Embedding Property, JEP)**.** Let $\mathcal{K}$ be an MAEC. We say that $\mathcal{K}$ satisfies *Joint Embedding Property* (for short *JEP*) if and only if for every $M_1, M_2 \in \mathcal{K}$ there exist $N \in \mathcal{N}$ and $\mathcal{K}$-embeddings $f_i : M_i \to N$ $(i \in \{1,2\})$.

Remark 2.6 (Monster Model)**.** If $\mathcal{K}$ is an MAEC which satisfies AP and JEP and has large enough models, then we can construct a large enough model $\mathbb{M}$ (which we call a *Monster Model*) which is homogeneous –i.e., every isomorphism between two $\mathcal{K}$-substructures of $\mathbb{M}$ can be extended to an automorphism of $\mathbb{M}$– and also universal –i.e., every model with density character $< dc(\mathbb{M})$ can be $\mathcal{K}$-embedded into $\mathbb{M}$.

Definition 2.7 (Galois type)**.** Under the existence of a monster model $\mathbb{M}$ as in remark 2.6, for all $\overline{a} \in \mathbb{M}$ and $N \prec_{\mathcal{K}} \mathbb{M}$, we define ga-tp$(\overline{a}/N)$ (the *Galois type of* $\overline{a}$ *over* N) as the orbit of $\overline{a}$ under $Aut(\mathbb{M}/N) := \{f \in Aut(\mathbb{M}) : f \upharpoonright N = id_N\}$. We denote the space of Galois types over a model $M \in \mathcal{K}$ by ga-S(M).

Throughout this paper, we assume the existence of a homogenous and universal monster model as in remark 2.6.

Definition 2.8 (Distance between types)**.** Let $p, q \in$ ga-$S(M)$. We define $d(p,q) := \inf\{d(\overline{a}, \overline{b}) : \overline{a}, \overline{b} \in \mathbb{M}, \overline{a} \models p, \overline{b} \models q\}$, where $\lg(\overline{a}) = \lg(\overline{b}) =: n$ and $d(\overline{a}, \overline{b}) := \max\{d(a_i, b_i) : 1 \leq i \leq n\}$.

Definition 2.9 (Continuity of Types)**.** Let $\mathcal{K}$ be an MAEC and consider $(a_n) \to a$ in $\mathbb{M}$. We say that $\mathcal{K}$ satisfies *Continuity of Types Property*[2] (for short, *CTP*), if and only if, if ga-tp$(a_n/M) =$ ga-tp(a_0/M) for all $n < \omega$ then ga-tp$(a/M) =$ ga-tp(a_0/M).

Remark 2.10. In general, distance between types d (see Definition 2.8) is just a pseudo-metric. But it is straightforward to see that the fact that d is a metric is equivalent to CTP.

Throughout this paper, we will assume our MAECs satisfy the CTP (so, distance between types is in fact a metric).

3. Existence of d-limit models in MAECs

In this section, we adapt one of the existing notions of limit models (see [GrVa]) to the metric context. This still leaves open the many variants of the notion that have recently been used by Shelah in NIP contexts (see for instance [Sh900]), or in strictly stable first order contexts.

Definition 3.1 (Universality)**.** Let $\mathcal{K}$ be an MAEC with CTP and $N \prec_{\mathcal{K}} M$. We say that M is λ-*universal* over N iff for every $N' \succ_{\mathcal{K}} N$ with density character λ there exists a $\mathcal{K}$-embedding $f : N' \to M$ such that $f \restriction N = id_N$. We say that M is *universal* over N if M is $dc(M)$-universal over N.

The following lemma will be useful later: it provides relative saturation criteria by iterating ω-many times dense relative saturation.

Lemma 3.2. *Suppose that we have an increasing $\prec_{\mathcal{K}}$-chain of models $(N_n : n < \omega)$ such that N_{n+1} realizes a dense subset of* ga-$S(N_n)$. *Then, every type in* ga-$S(N_0)$ *is realized in* $N_\omega := \overline{\bigcup_{n<\omega} N_n}$.

Proof. See [ViZa1, Lemma 1.18]. □

Definition 3.3. Let $M, N \in \mathcal{K}$ be such that $M \prec_{\mathcal{K}} N$. We say that N is μ-**d**-universal over N iff for every $N' \succ_{\mathcal{K}} N$ such that $dc(N) = \mu$ we have that there exists a $\prec_{\mathcal{K}}$-embedding $f : N' \to M$ which fixes pointwise M. We say that N is **d**-universal over M iff N is $dc(M)$-**d**-universal over M.

[2]This property is also called *Perturbation Property in* [Hi]

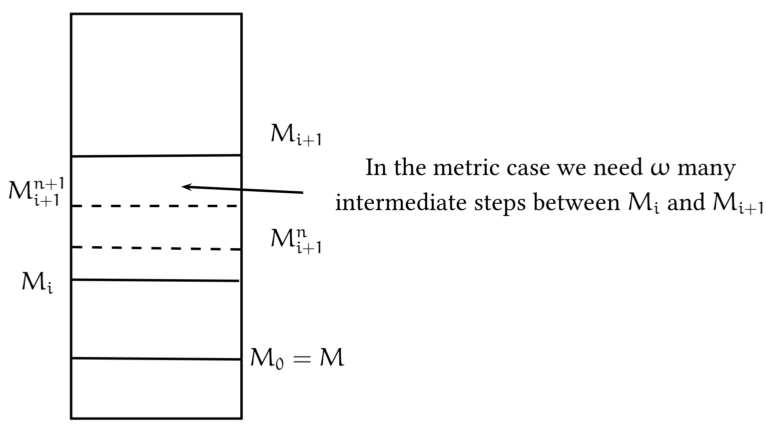
We drop the prefix **d** if it is clear that we are working in a metric setting.

Definition 3.4. Let $M, N \in \mathcal{K}$ be such that $M \prec_{\mathcal{K}} N$, where $dc(M) = \mu$. We say that N is (μ, θ)-**d**-limit over N iff there exists an increasing and continuous $\prec_{\mathcal{K}}$-chain $(M_i : i < \theta)$ such that $\overline{\bigcup_{i<\theta} M_i} = N$, $dc(M_i) = \mu$ for every $i < \theta$ and also M_{i+1} is μ-**d**-universal over M_i.

Definition 3.5. We say that $\mathcal{K}$ is μ-**d**-stable iff for every $M \in \mathcal{K}$ such that $dc(M) \leq \mu$ we have that $dc(\text{ga-S}(M)) \leq \mu$

We now prove the existence of universal extensions in the setting of Metric Abstract Elementary Classes. We point out that this is an adaptation of the proof of the existence of universal extensions over a given model M in the setting of Abstract Elementary Classes (see [GrVa]). In that proof, under μ-stability, we can consider an increasing and continuous $\mathcal{K}$-chain $\langle M_i : i < \mu \rangle$ such that $M_0 := M$ and where M_{i+1} realizes every Galois-type in ga-S(M_i). So, $\bigcup_{i<\mu} M_i$ is universal over M. But in this setting, we cannot consider directly from μ-**d**-stability that M_{i+1} realizes every type in ga-S(M_i). But we use Lemma 3.2 in a suitable way for guaranteeing that requirement.

Proposition 3.6 (Existence of universal extensions)**.** *Let $\mathcal{K}$ be a MAEC μ-**d**-stable with AP and CTP. Then for all $\mathcal{M} \in \mathcal{K}$ such that $dc(\mathcal{M}) = \mu$ there exists $\mathcal{M}^* \in \mathcal{K}$ universal over $\mathcal{M}$. such that $dc(\mathcal{M}^*) = \mu$*



Proof. The proof follows almost along the same lines as the proof of existence of universal models in usual AECs (see Claim 2.9 of [GrVa] and Claim 1.15.1 of [Sh600]); that is, by trying to capture realizations of types along the construction in a coherent way, and building the universal extension as a union of a chain (we do not repeat all the details of the proof, but point out the differences).

In our metric setting, we need to be careful with the way we realize the types along the construction: although this cannot be done in an immediate way in each successor stage as in [GrVa], lemma 3.2 provides the realizations we need of dense subsets of the typespace in ω many steps.

We construct an increasing and continuous $\prec_{\mathcal{K}}$-chain of models $\langle M_i : i < \mu \rangle$ such that $M_0 := M$, M_{i+1} is the completion of the union of a resolution $(M^n_{i+1} : n < \omega)$ where $M^0_{i+1} := M_i$ M^{n+1}_{i+1} realizes a dense subset of ga-$S(M^n_{i+1})$ and $dc(M^n_{i+1}) = \mu$ for every $n < \omega$. This is possible by μ-$\mathbf{d}$-stability of $\mathcal{K}$. Take $M^* := \overline{\bigcup_{i<\mu} M_i}$. M^* turns out to be universal over M — by the same argument as in Claim 2.9 of [GrVa]. $\square_{\text{Prop. 3.6}}$

Corollary 3.7. *Let $\mathcal{K}$ be a MAEC μ-$\mathbf{d}$-stable with AP. Then for all $M \in \mathcal{K}$ such that $dc(M) = \mu$ there exists $M^* \in \mathcal{K}$ limit over M such that $dc(M^*) = \mu$*

Proof. Iterate the construction given in proposition 3.6. $\square_{\text{Cor. 3.7}}$

4. Smooth independence in MAECs.

In this section, also we define the notion of *smooth independence* for MAECs with CTP and AP, and state some of its properties. For a more complete analysis of this independence notion, see [ViZa1, Za].

Definition 4.1 (ε-splitting and $\downarrow^{\varepsilon}$)**.** Let $N \prec_{\mathcal{K}} M$ and $\varepsilon > 0$. We say that ga-tp(a/M) *ε-splits* over N iff there exist N_1, N_2 with $N \prec_{\mathcal{K}} N_1, N_2 \prec_{\mathcal{K}} M$ and $h : N_1 \approx_N N_2$ such that $\mathbf{d}(\text{ga-tp}(a/N_2), h(\text{ga-tp}(a/N_1)) \geq \varepsilon$. We use $a \downarrow^{\varepsilon}_N M$ to denote the fact that ga-tp(a/M) does not ε-split over N,

Definition 4.2. Let $N \prec_{\mathcal{K}} M$. Fix $\mathcal{N} := \langle N_i : i < \sigma \rangle$ a resolution of N. We say that a is *smoothly independent* from M over N relative to $\mathcal{N}$ (denoted by $a \downarrow^{\mathcal{N}}_N M$) iff for every $\varepsilon > 0$ there exists $i_\varepsilon < \sigma$ such that $a \downarrow^{\varepsilon}_{N_{i_\varepsilon}} M$.

In the following lines, we provide a list of properties -without proofs- of smooth independence. For the proofs, see [ViZa1, Za].

Proposition 4.3 (Monotonicity of non-ε-splitting)**.** *Let $M_0 \prec_{\mathcal{K}} M_1 \prec_{\mathcal{K}} M_2 \prec_{\mathcal{K}} M_3$. If $a \downarrow^{\varepsilon}_{M_0} M_3$ then $a \downarrow^{\varepsilon}_{M_1} M_2$.*

Proposition 4.4 (Monotonicity of smooth-independence)**.** *Let $M_0 \prec_{\mathcal{K}} M_1 \prec_{\mathcal{K}} M_2 \prec_{\mathcal{K}} M_3$. Fix $\mathcal{M}_k := \langle M^k_i : i < \sigma_k \rangle$ a resolution of M_k ($k = 0, 1$), where $\mathcal{M}_0 \subseteq \mathcal{M}_1$. If $a \downarrow^{\mathcal{M}_0}_{M_0} M_3$ then $a \downarrow^{\mathcal{M}_1}_{M_1} M_2$.*

Lemma 4.5 (Stationarity (1)). *Suppose that* $N_0 \prec_{\mathcal{K}} N_1 \prec_{\mathcal{K}} N_2$ *and* N_1 *is universal over* N_0. *If* $\text{ga-tp}(a/N_1) = \text{ga-tp}(b/N_1)$, $a \downarrow^{\varepsilon}_{N_0} N_2$ *and* $b \downarrow^{\varepsilon}_{N_0} N_2$, *then* $\mathbf{d}(\text{ga-tp}(a/N_2), \text{ga-tp}(b/N_2) < 2\varepsilon$.

Proposition 4.6 (Stationarity (2)). *If* $N \prec_{\mathcal{K}} M \prec_{\mathcal{K}} M'$, M *is universal over* N, $\mathcal{N} := \langle N_i : i < \sigma \rangle$ *a resolution of* N *and* $p := \text{ga-tp}(a/M) \in \text{ga-S}(M)$ *is a Galois type such that* $a \downarrow^{\mathcal{N}}_{N} M$, *then there exists an unique extension* $p^* \supset p$ *over* M' *which is independent (relative to* $\mathcal{N}$*) from* M' *over* N.

Proposition 4.7 (Continuity of smooth-independence). *Let* $(b_n)_{n<\omega}$ *be a convergent sequence and* $b := \lim_{n<\omega} b_n$. *If* $b_n \downarrow^{\mathcal{N}}_{N} M$ *for every* $n < \omega$, *then* $b \downarrow^{\mathcal{N}}_{N} M$.

Proposition 4.8 (stationarity (3)). *Let* $M_0 \prec_{\mathcal{K}} M \prec_{\mathcal{K}} N$ *be such that* M *is a* (μ, σ)*-limit model over* M_0, *witnessed by* $\mathcal{M} := \{M_i : i < \sigma\}$. *If* $a, b \downarrow^{\mathcal{M}}_{M} N$ *and* $\text{ga-tp}(a/M) = \text{ga-tp}(b/M)$, *then* $\text{ga-tp}(a/N) = \text{ga-tp}(b/N)$.

Proposition 4.9 (Transitivity). *Let* $M_0 \prec_{\mathcal{K}} M_1 \prec_{\mathcal{K}} M_2$ *be such that* M_0 *is a limit model over some* $M' \prec_{\mathcal{K}} M_0 \prec_{\mathcal{K}} M_1$ *(witnessed by* $\mathcal{M}_0$*) and* M_1 *is a limit model over* M_0 *(witnessed by* $\mathcal{M}_1'$*). Let* $\mathcal{M}_1 := \mathcal{M}_0 \cup \mathcal{M}_1'$, *so* $\mathcal{M}_0 \subset \mathcal{M}_1$. *Then* $a \downarrow^{\mathcal{M}_0}_{M_0} M_2$ *iff* $a \downarrow^{\mathcal{M}_0}_{M_0} M_1$ *and* $a \downarrow^{\mathcal{M}_1}_{M_1} M_2$.

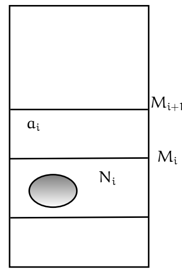
5. Smooth towers.

Throughout this section, we assume that all our models have density character μ, all orderings denoted by I, I', I_β, etc. have cardinality μ as well, and $cf(I) = cf(I') = cf(I_\beta) > \omega$, unless we state the contrary.

Assumption 5.1 (superstability). *For every* a *and every increasing and continuous* $\prec_{\mathcal{K}}$*-chain of models* $\langle M_i : i < \sigma \rangle$ *and* $\mathcal{M}_j$ *a resolution of* M_j *(*$j < \sigma$*):*

1. *If* $p \upharpoonright M_i \downarrow^{\mathcal{M}_0}_{M_0} M_i$ *for all* $i < \sigma$, *then* $p \downarrow^{\mathcal{M}_0}_{M_0} \overline{\bigcup_{i<\sigma} M_i}$.
2. *if* $cf(\sigma) > \omega$, *there exists* $j < \sigma$ *such that* $a \downarrow^{\mathcal{M}_j}_{M_j} \bigcup_{i<\sigma} M_i$.
3. *if* $cf(\sigma) = \omega$, *there exists* $j < \sigma$ *such that* $a \downarrow^{\varepsilon}_{M_j} \overline{\bigcup_{i<\sigma} M_i}$.

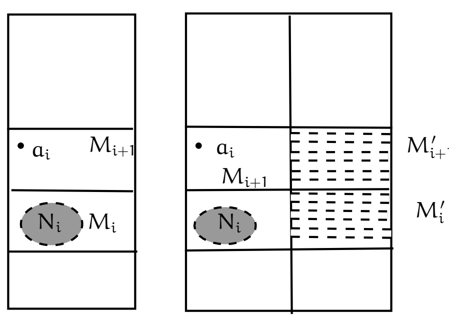
Definition 5.2 (smooth towers). Smooth tower Let I be a well-ordering, $\mathfrak{M} := (M_i : i \in I)$ be an $\prec_{\mathcal{K}}$-increasing chain, $\overline{a} := (a_i : i \in I)$, $\mathfrak{N} := (N_i : i < \sigma)$ be a sequence of models in $\mathcal{K}$, $\mathcal{M} := (\mathcal{M}_j : j \in I)$ be a sequence of resolutions $\mathcal{M}_j$ of M_j ($j \in I$) and $\mathcal{N} := (\mathcal{N}_j : j \in I)$ be a sequence of resolutions $\mathcal{N}_j$ of N_j ($j \in I$). We say that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ is a *smooth tower* (shortly, *s-tower*) iff for every $i \in I$ we have that M_i is a (μ, σ)-limit model over N_i for some $\sigma < \mu^+$, $a_i \in M_{i+1} \setminus M_i$ and $a_i \downarrow^{\mathcal{N}_i}_{N_i} M_i$.



Roughly speaking, an s-tower is composed by a $\prec_{\mathcal{K}}$-increasing (not necessarily continuous) chain of models $\mathfrak{M} := (M_i : i \in I)$ and a collection of $\mathcal{K}$-submodels $\mathfrak{N} := (N_i : i \in I)$ such that each M_i is a (μ, σ)-limit model over N_i (for some $\sigma < \mu^+$) which codify a smooth independence of the elements a_i taken in the s-tower (i.e., $a_i \mathop{\smile\!\!\!\!\!\!\mid}_{N_i}^{\mathcal{N}_i} M_i$).

Definition 5.3 (Extension of s-towers)**.** Let $I \leq I'$ be well-orderings, $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N}) \in \mathcal{K}_{\mu,I}$ and $(\mathfrak{M}', \overline{a}', \mathfrak{N}', M', N') \in \mathcal{K}_{\mu,I'}$. We say that $(\mathfrak{M}', \overline{a}', \mathfrak{N}', M', N')$ extends $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ (denoted by $(\mathfrak{M}', \overline{a}', \mathfrak{N}', M', N') \geq (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$) iff for every $i \in I$:

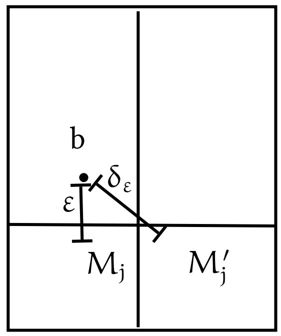
(1) M'_i is a proper universal model over M_i
(2) $\mathcal{M}_i \subset \mathcal{M}'_i$.
(3) $a_i = a'_i$
(4) $N_i = N'_i$
(5) $\mathcal{N}_i = \mathcal{N}'_i$



6. d-REDUCED TOWERS.

Definition 6.1. An I-tower $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ is said to be a d-*reduced tower* iff there exists a mapping $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\delta(\alpha \cdot \varepsilon) = \alpha \cdot \delta(\varepsilon)$ for every $\alpha, \varepsilon > 0$, in such a way that if $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})' \geq (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ then for every $j \in I$ and every $\varepsilon > 0$, if $b \in \overline{\bigcup_{i \in I} M_i}$ and $d(b; M'_j) < \delta_\varepsilon := \delta(\varepsilon)$ then $d(b; M_j) < \varepsilon$.

$(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N}) \qquad (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})'$



Proposition 6.2 (Density of reduced towers). *Given* $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ *an* I*-tower, there exists* $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})' \geq (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ *such that* $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})'$ *is a reduced tower.*

Proof. Suppose the proposition is false. Let $\langle(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\alpha : \alpha < \mu^+\rangle$ be an $\leq$-increasing chain of towers such that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^{\alpha+1} \geq (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\alpha$ witnesses that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\alpha$ is not a reduced tower, where $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^0 := (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$.

Let $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^{\mu^+}$ be the completion of the union of $\langle(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\alpha : \alpha < \mu^+ : \alpha < \beta < \mu^+\rangle$. Fix $\varepsilon > 0$ and $b \in \overline{\bigcup_{i\in I} M_i^{\mu^+}}$; define $i_\varepsilon(b) := \min\{i \in I :$ there exists $b' \in \bigcup_{j\leq i} M_i^{\mu^+}$ such that $d(b, b') < \varepsilon\}$ and $\zeta_\varepsilon(b) := \min\{\zeta < \mu^+ :$ there exists $b' \in M^\zeta_{i_\varepsilon(b)}$ such that $d(b', b) < \varepsilon\}$. Let $E := \{\zeta < \mu^+ :$ for all $b \in \overline{\bigcup_{i\in I} M_i^\zeta}$, $\zeta_\varepsilon(b) < \zeta\}$. E is a stationary subset of λ^+.

Notice that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^{\zeta+1}$ witnesses that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\zeta$ is not a d-reduced tower; i.e., there exists $\varepsilon > 0$ such that for every linear mapping $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ there exists $b \in \overline{\bigcup_{i\in I} M_i^\zeta}$ such that for some $j \in I$ $d(b, M_j^{\zeta+1}) < \delta(\varepsilon)$ but $d(b, M_j^\zeta) = d(b, M_j^\zeta) \geq \varepsilon$. Since $b \in \overline{\bigcup_{i\in I} M_i^\zeta}$, there exists $(b_n) \in \bigcup_{i\in I} M_i^\zeta$ such that $(b_n) \to b$. So, there exists $N < \omega$ such that $d(b_N, b) < \varepsilon/2$. Let $j \in I$ such that $b_N \in M_j^\zeta$, therefore $i_{\varepsilon/2}(b) \leq j$. Notice that $\zeta_{\varepsilon/2}(b) < \zeta$, since $\zeta \in E$. Since $b_N \in M^\zeta_{i_{\varepsilon/2}(b)}$, there exists $(c_n) \in \bigcup_{\alpha<\zeta} M^\alpha_{i_\varepsilon(b)}$ such that $(c_n) \to b_N$. Let $M < \omega$ be such that $d(c_M, b_N) < \varepsilon/2$, so $d(c_M, b) \leq d(c_M, b_N) + d(b_N, b) < \varepsilon$. Let $\alpha < \zeta$ be such that $c_M \in M^\alpha_{i_{\varepsilon/2}(b)}$. Notice that $d(b; M_j^\zeta) \leq d(b; M^\zeta_{i_{\varepsilon/2}(b)}) \leq d(b, M^\alpha_{i_{\varepsilon/2}(b)}) < \varepsilon$ (contradiction). $\square_{\text{Prop. 6.2}}$

Proposition 6.3. *Let* $\langle (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})_i : i < \beta \rangle$ *be an* $\leq$*-increasing sequence of* d*-reduced towers. Then the completion of its union is a* d*-reduced tower.*

Proof. Let $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})_\beta$ be the completion of the union of $\langle (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})_i : i < \beta \rangle$ and let $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})' \geq (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})_\beta$.

Let $\varepsilon > 0$ and $K < \omega$ be such that $2\delta_\varepsilon/K < \varepsilon$.

Let $\varepsilon' := \varepsilon - 2\delta_\varepsilon/K > 0$ and $\varepsilon'' := \min\{\delta_{5\varepsilon'/6}/10, \delta_\varepsilon/K\}$.

Let $b \in \overline{\bigcup_{i\in I} M_i^\beta}$ be such that $d(f(b); M_j') < \delta_{5\varepsilon'/6}$. Since $b \in \overline{\bigcup_{i\in I} M_i^\beta}$, then there exists a sequence $(a_n) \in \bigcup_{i\in I} M_i^\beta$ such that $(a_n) \to b$. So, there exists $n_\varepsilon < \omega$ such that

$$d(a_{n_\varepsilon}, b) < \varepsilon''$$

Let $k_\varepsilon \in I$ be such that $a_{n_\varepsilon} \in M_{k_\varepsilon}^\beta$. By definition of $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})_\beta$, $M_{k_\varepsilon}^\beta := \overline{\bigcup_{\alpha(k_\varepsilon)\leq\alpha<\beta} M_{k_\varepsilon}^\alpha}$, hence there exists $(c_m) \in \bigcup_{\alpha(k_\varepsilon)\leq\alpha<\beta} M_{k_\varepsilon}^\alpha$ such that $(c_m) \to a_{n_\varepsilon}$. Therefore, there exists $m_\varepsilon < \omega$ such that

$$d(c_{m_\varepsilon}, a_{n_\varepsilon}) < \varepsilon''$$

Let $\alpha_\varepsilon < \beta$ be such that $c_{m_\varepsilon} \in M_{k_\varepsilon}^{\alpha_\varepsilon}$.

Let $y \in M_j'$ be such that $d(b, y) < \delta_{5\varepsilon'/6}$ (this exists because $d(b, M_j') < \delta_{5\varepsilon'/6}$). Notice that

$$\begin{aligned}
d(c_{m_\varepsilon}, y) &\leq d(c_{m_\varepsilon}, a_{n_\varepsilon}) + d(a_{n_\varepsilon}, b) + d(b, y) \\
&< 2\varepsilon'' + \delta_{5\varepsilon'/6} \\
&\leq 2\delta_{5\varepsilon'/6}/10 + \delta_{5\varepsilon'/6} \\
&= \frac{6}{5}\delta_{5\varepsilon'/6} \\
&= \delta_{\frac{6}{5}\cdot\frac{5\varepsilon'}{6}} \\
&= \delta_{\varepsilon'}
\end{aligned}$$

Since $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})' \geq (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})_{\alpha_\varepsilon}$ and

$$\begin{aligned}
d(c_{m_\varepsilon}, y) &< \delta_{\varepsilon'} \\
&= \delta_{\varepsilon - 2\delta_\varepsilon/K}
\end{aligned}$$

then $d(c_{m_\varepsilon}; M_j') < \delta_{\varepsilon-2\delta_\varepsilon/K}$; since $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})_{\alpha_\varepsilon}$ is d-reduced then $d(c_{m_\varepsilon}; M_j^{\alpha_\varepsilon}) <= \varepsilon - 2\delta_\varepsilon/K$.

Let $x \in M_j^{\alpha_\varepsilon}$ be such that $d(c_{m_\varepsilon}, x) < \varepsilon - 2\delta_\varepsilon/K$. Notice that

$$\begin{aligned} d(b,x) &\leq d(b,a_{n_\varepsilon}) + d(a_{n_\varepsilon}, c_{m_\varepsilon}) + d(c_{m_\varepsilon}, x) \\ &< 2\varepsilon'' + (\varepsilon - 2\delta_\varepsilon/K) \\ &\leq 2\delta_\varepsilon/K + (\varepsilon - 2\delta_\varepsilon/K) \\ &= \varepsilon \end{aligned}$$

Since $M_j^{\alpha_\varepsilon} \prec_{\mathcal{K}} M_j^\beta$, then $d(b, M_j^\beta) < \varepsilon$. $\square_{\text{Prop. 6.3}}$

Fact 6.4. *If* $M, N \in \mathcal{K}$ *are* λ*-d-saturated models of density character* λ^+ *then* $M \approx N$.

Proof. By a standard back and forth argument, as in discrete AECs. $\square_{\text{Fact 6.4.}}$

Proposition 6.5. *Let* $\mathcal{K}$ *be* μ^+*-d-categorical. Then every d-reduced tower of density character* μ *is continuous.*

Proof. Suppose there is a d-reduced $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ which is not continuous; i.e., there exists a limit element $\delta \in I$ and $b \in M_\delta$ such that $b \notin \overline{\bigcup_{i<\delta} M_i}$. By density property (Prop. 6.2), we can build an $\leq$-increasing sequence of d-reduced towers $\langle (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})_\alpha : \alpha < \mu^+ \rangle$ such that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})_0 := (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N}) \restriction \delta$. Define
$(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})_{\mu^+} := \overline{\bigcup_{\alpha<\mu^+} (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})_\alpha}$.
Suppose $M := \overline{\bigcup_{i<\delta} M_i^{\mu^+}}$ realizes ga-tp$(b/\overline{\bigcup_{i<\delta} M_i})$, so there exist $b' \in M$ and $f \in Aut(\mathbb{M}/\overline{\bigcup_{i<\delta} M_i})$ such that $f(b') = b$. Define
$(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})' := f[(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})_{\mu^+}]$.
Notice that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})_0 \leq (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})'$ and $b \in \overline{\bigcup_{i<\delta} M'_i}$. Let $\varepsilon := d(b; \overline{\bigcup_{i<\delta} M_i}) > 0$. Since $b \in \overline{\bigcup_{i<\delta} M'_i}$ there exists $(b_n) \in \bigcup_{i<\delta} M'_i$ such that $(b_n) \to b$. We can assure that there exists $N < \omega$ such that $d(b_N, b) < \delta(\varepsilon)/2$. Let $i < \delta$ be such that $b_N \in M'_i$. Since $M'_i := f[\overline{\bigcup_{\alpha<\mu^+} M_i^\alpha}] = \overline{\bigcup_{\alpha<\mu^+} f[M_i^\alpha]}$, then there exists $(c_m) \in \bigcup_{\alpha<\mu^+} f[M_i^\alpha]$ such that $(c_n) \to b_N$. We can find $M < \omega$ such that $d(c_M, b_N) < \delta(\varepsilon)/2$, so $d(c_M, b) \leq d(c_M, b_N) + d(b_N, b) < \delta(\varepsilon)$. Notice that $f[(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})_\alpha] \geq (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})_0$ and $d(b; f[M_i^\alpha]) \leq d(b, c_M) < \delta(\varepsilon)$; since $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ is a d-reduced tower then
$d(b; \overline{\bigcup_{i<\delta} M_i}) \leq d(b; M_i) < \varepsilon$ (contradiction).

So, M is not a μ-d-saturated model of density character μ^+. By using Ehrenfeucht-Mostowski models, there exists a model $N \in \mathcal{K}$ μ-d-saturated of density character μ^+. Since $M \not\approx N$, this contradicts μ^+-d-categoricity.

$\square_{\text{Prop. 6.5}}$

7. Full-relativeness of s-towers

Definition 7.1 (strong type). Let M be a σ-limit model

(1) $$\mathfrak{St}(M) := \left\{ (p, N) : \begin{array}{l} N \prec_{\mathcal{K}} M \\ N \text{ is a } \theta\text{-limit model} \\ M \text{ is universal over } N \\ p \in \text{ga-S}(M) \text{ is non-algebraic} \\ \text{and } p \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_N^{\mathcal{N}} M \\ \text{for some resolution } \mathcal{N} \text{ of } N. \end{array} \right\}$$

(2) Two strong types $(p_l, N_l) \in \mathfrak{St}(M_l)$ ($l \in \{1, 2\}$) are *parallel* (which we denote by $(p_1, N_1) \parallel (p_2, N_2)$) iff for every $M' \succ_{\mathcal{K}} M_1, M_2$ with density character μ, there exists $q \in \text{ga-S}(M')$ which extends both p_1 and p_2 and $q \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_{N_l}^{\mathcal{N}_l} M'$ ($l \in \{1, 2\}$) (where $\mathcal{N}_l$ is the resolution of N_l which satisfies $p_i \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_{N_l}^{\mathcal{N}_l} M_l$).

Assumption 7.2. *Through this subsection, assume that* I *is a well ordered set which has a cofinal sequence* $(i_\alpha : \alpha < \theta)$*, where* $\text{cf}(\theta) > \omega$*.*

Definition 7.3 (Metric s-Towers). An s-tower $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ is called a *metric s-tower* if the resolution witnessing that M_i is a (μ, σ)-limit model over N_i is spread-out. A spread-out resolution $\mathcal{M}$ of M is a resolution where for every γ, $M^{\gamma+1}$ is an ω_1-limit over M^γ.

Definition 7.4 (Full relative s-towers). Let $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ be a s-tower indexed by I. Let $(M_i^\gamma : \gamma < \sigma)$ be a sequence which witnesses that M_i is a (μ, σ)-limit model. We say that $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ is a relative s-tower with respect to $(M_i^\gamma)_{i \in I, \gamma < \sigma}$ iff for every $i_\alpha \leq i < i_{\alpha+1}$ and $(p, M_i^\gamma) \in \mathfrak{St}(M_i)$ there exists $i \leq j < i_{\alpha+1}$ such that $(p, M_i^\gamma) \parallel (\text{ga-tp}(a_j/M_j), N_j)$.

Proposition 7.5. *Suppose that for every* $\alpha < \theta$ *there are* $\mu \cdot \omega$ *many elements between* i_α *and* $i_{\alpha+1}$*. Let* $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ *be a full relative s-tower with respect to* $(M_i^\gamma)_{i \in I, \gamma < \sigma}$*. Then* $M := \overline{\bigcup_{i \in I} M_i}$ *is a limit model over* M_{i_0}*.*

Proof. It is enough to prove that $M_{i_{\alpha+1}}$ is universal over M_{i_α}. Let $p := \text{ga-tp}(a/M_{i_\alpha}) \in \text{ga-S}(M_{i_\alpha})$ and $\varepsilon > 0$. So, by assumption 5.1 there exists $\gamma := \gamma_\varepsilon < \sigma$ such that $a \mathop{\smile\!\!\!\!\!\!\,|}\nolimits_{M_{i_0}^{\gamma_\varepsilon}}^{\varepsilon} M_{i_0}$.

By construction, $M_{i_\alpha}^{\gamma+1}$ is a (μ, ω_1)-limit model over $M_{i_\alpha}^\gamma$. Let $(M_i^* : i < \omega_1)$ be a resolution which witnesses that.

Consider $q := p \upharpoonright M_{i_\alpha}^{\gamma+1}$, so by assumption 5.1 there exists $i < \omega_1$ such

that $q \downfree^{\mathcal{M}_i^*}_{M_i^*} M^{\gamma+1}_{i_\alpha}$. By extension over universal models (proposition 4.6, notice that $M^{\gamma+1}_{i_\alpha}$ is universal over M_i^*), there exists $q^* \in \text{ga-S}(M_{i_\alpha})$ an extension of q such that $q^* \downfree^{\mathcal{M}_i^*}_{M_i^*} M_{i_\alpha}$. So, $(q^*, M_i^*) \in \mathfrak{St}(M_{i_\alpha})$. By full relativeness of $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$, there exists $i_\alpha \leq j_1 < i_{\alpha+1}$ such that $(q^*, M_i^*) \parallel (\text{ga-tp}(a_j/M_j), N_j)$. Therefore, $q^* = \text{ga-tp}(a_j/M_{i_\alpha})$ and so q^* is realized in M_{j_1}.

By monotonicity of non-ε-splitting, we have that p does not ε-split over M_i^* (since p does not ε-split over $M^\gamma_{i_\gamma}$ and $M^\gamma_{i_\alpha} \prec_{\mathcal{K}} M_i^*$); i.e. $p \downfree^{\varepsilon}_{M_i^*} M_{i_\alpha}$. Since $q^* \downfree^{\mathcal{M}_i^*}_{M_i^*} M_{i_\alpha}$, then $q^* \downfree^{\varepsilon}_{M_i^*} M_{i_\alpha}$ (by monotonicity of non-ε-splitting, proposition 4.3).

Also, since $q = p \upharpoonright M^{\gamma+1}_{i_\alpha}$ and $q^* \supset q$, then $q^* \upharpoonright M^{\gamma+1}_{i_\alpha} = p \upharpoonright M^{\gamma+1}_{i_\alpha}$. Notice that $M^{\gamma+1}_{i_\alpha}$ is universal over M_i^*.

Since $p, q^* \downfree^{\varepsilon}_{M_i^*} M_{i_\alpha}$, by a weak version of stationarity (proposition 4.5) we have that $\mathbf{d}(p, q^*) < 2\varepsilon$. Therefore, M_{j_1} realizes a dense subset of $\text{ga-S}(M_{i_\alpha})$.

Doing a similar argument, we can construct an increasing sequence $(j_n : n < \omega)$ in I (where $j_0 := i_\alpha$) such that $i_\alpha \leq j_n < i_{\alpha+1}$, where $M_{j_{n+1}}$ realizes a dense subset of $\text{ga-S}(M_{j_n})$.

Therefore, by lemma 3.2 we have that $M^* := \overline{\bigcup_{n<\omega} M_{j_n}} \prec_{\mathcal{K}} M_{i_{\alpha+1}}$ realizes every type over $M_{j_0} = M_{i_\alpha}$, so $M_{i_{\alpha+1}}$ does. $\square_{\text{Prop. 7.5}}$

The following fact is proved in a similar way like the discrete case (see [GrVaVi]). For the sake of completeness, we give a proof of this result.

Proposition 7.6. *If* $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N}) \in \mathcal{K}^*_{\mu,I}$, *there exists* $(\mathfrak{M}', a, \mathfrak{N}, \mathcal{M}', \mathcal{N}) > (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ *in* $\mathcal{K}^*_{\mu,I}$ *such that for every limit* $i \in I$, M'_i *is a* (μ, μ)*-limit over* $\overline{\bigcup_{j<i} M_j}$

Proof. First, we construct by induction on $i \in I$ a model $M_i^+ \succ_{\mathcal{K}} M_i$ and a directed system $(f_{i,j} : i < j \in I)$ of $\prec_{\mathcal{K}}$-embeddings (as in the discrete AEC case, one may prove that the "union axioms" for metric AEC also hold for directed systems) such that $f_{i,j} : M_i^+ \to M_j^+$ and $f_{i,j} \upharpoonright M_i = id_{M_i}$.

Suppose $(M_k^+ : k \leq i)$ and $(f_{k,l} : k < l \leq i)$ are constructed. We give the construction of M^+_{i+1} and $f_{i,i+1}$. The construction of $f_{j,i+1}$ $(j < i)$

are given by definition of directed system. Let M^*_{i+1} be a limit model over M^+_i and M_{i+1}. Since $a_{i+1}\mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathcal{N}_{i+1}}_{N_{i+1}}M_{i+1}$ and M_{i+1} is universal over N_{i+1} (by definition of s-tower), by the extension property (proposition 4.6) and invariance of smooth independence there exists $f \in \mathrm{Aut}(\mathbb{M}/M_{i+1})$ such that $a_{i+a}\mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathcal{N}_{i+1}}_{N_{i+1}}f[M^*_{i+1}]$. Define $M^+_{i+1} := f[M^*_{i+1}]$ and $f_{i,i+1} := f \restriction M^+_i$.

For limit $i \in I$, first take the directed limit of $(M^+_k : k \leq i)$ and $(f_{k,l} : k < l \leq i)$ and then consider M^+_i a limit model over this directed limit and (μ, μ)-limit over $\overline{\bigcup_{j<i} f_{j,i}[M^+_j]}$.

Fix $j \in I$. Let $f_{j,\sup(I)}$ and $M'_{j,\sup(I)}$ be the respective directed limit of this directed system. Notice that $f_{j,\sup(I)} \restriction M_j = id_{M_j}$. Define $M'_j := f_{j,\sup(I)}[M^+_j]$.

Notice that the s-tower $(\mathfrak{M}', \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ defined in this way satisfies the requirements of the proposition.

$\square_{\text{Prop. 7.6}}$

Lemma 7.7 (Weak Full Relativeness). *Given* $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N}) \in \mathcal{K}^*_{\mu,I_n}$, *there exists* $(\mathfrak{M}', a, \mathfrak{N}, \mathcal{M}', \mathcal{N}) > (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})$ *in* $\mathcal{K}^*_{\mu,I_{n+1}}$ *such that for every* $(p, N) \in \mathfrak{St}(M_i)$ *(where* $i \in I_n$ *and* $i_\alpha \leq i < i_{\alpha+1}$*) there exists* $i \leq j < i_{\alpha+1}$ *such that* $(\text{ga-tp}(a_j/M'_j), N_j) \parallel (p, N)$.

Proof. Let $M'_{i_{\alpha+1}}$ be a (μ, μ)-limit model over $\overline{\bigcup_{j<i_{\alpha+1},j\in I_n} M_j}$ (by proposition 7.6). Let $\langle M'_i : l \in I_{n+i}, i_\alpha+\mu\cdot n < l < \alpha + 1\rangle$ be an enumeration of a resolution which witnesses that $M'_{i_{\alpha+1}}$ is (μ, μ)-limit over $\overline{\bigcup_{j<i_{\alpha+1},j\in I_n} M_j}$.

Let $\mathfrak{S} := \{(p, N)^l_\alpha : i_\alpha + \mu \cdot n < l < i_{\alpha+1}\}$ be an enumeration of a dense subset of $\bigcup\{\mathfrak{St}(M_i) : i \in I_n, i_\alpha \leq i < i_{\alpha+1}\}$ (by μ-stability). Therefore, given $(p, N)^l_\alpha \in \mathfrak{S}$ there exists $i \in I_n$ such that $i_\alpha \leq i < i_{\alpha+1}$ such that $(p, N)^l_\alpha \in \mathfrak{St}(M_i)$. So $p^l_\alpha\mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathcal{N}^l_\alpha}_{N^l_\alpha}M_i$. Since by definition of strong type M_i is universal over N^l_α and $M_i \prec_{\mathcal{K}} M'_l$, by proposition 4.6 there exists $p^* \in \text{ga-S}(M'_l)$ which extends p^l_α and $p^*\mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathcal{N}^l_\alpha}_{N^l_\alpha}M'_l$. Notice that $M'_{\mathrm{succ}_{I_{n+1}}(l)}$ is universal over M'_l (by construction), then there exists $a_l \in M'_{\mathrm{succ}_{I_{n+1}}(l)}$ such that $a_l \models p^*$. Consider $N_l := N^l_\alpha$. So, $a_l\mathop{\smile\!\!\!\!\!\!|}\nolimits^{\mathcal{N}}_{N_l}M'_l$. The s-tower constructed in this way satisfies the requirements of the proposition.

$\square_{\text{Lemma 7.7}}$

8. Uniqueness of Limit Models

We now put together the material from the previous three sections and finish the proof of uniqueness of limit models in the categorical case for Metric Abstract Elementary Classes that have Amalgamation and Continuity of Types (MAEC + AP + CTP).

Part of the outline of the proof is inspired in the proof of the analogous results given by Grossberg, VanDieren and the first author of this paper in [GrVaVi]. There are however various deep differences in the lemmas that fill the outline, due to the difference in independence notions, in the notion of "reduced tower" and in the proof of continuity of reduced towers here. The metric context forces us to thread finely and deal with differences that are not visible in the usual (discrete) AEC context.

However, the results here follow a general outline of proof that already has a history in the proof of Uniqueness of Limit Models in "superstable" AECs - in this very particular sense, this paper is a contribution to the superstability of *metric* Abstract Elementary Classes where the types are orbital (AP) and are also endowed with a metric (CTP).

Proposition 8.1. *There is a* (μ, θ)-d-*limit model over* M *which is also a* (μ, ω)-d-*limit model over* M.

Proof. Consider a tower $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^0 \in \mathcal{K}_{\mu, I_0}$ such that $M^0_0 := M$. Suppose that we have constructed $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^n \in \mathcal{K}_{\mu, I_n}$. By lemma 7.7 and proposition 6.2, there exists an s-tower $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^n \leq (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^{n+1}$ which is reduced and also satisfies the properties given in lemma 7.7. At I_ω, consider $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\omega$ as the completion of the union of $\langle (\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^n : n \leq m < \omega \rangle$. By proposition 6.3, $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\omega$ is a reduced tower (and so continuous, by proposition 6.5).

Claim 8.2. $M^\omega_{i_\theta}$ *is a* (μ, ω)-d-*limit model witnessed by* $\{M^n_{i_\theta} : n < \omega\}$

Proof. By definition of $\leq$. $\square_{\text{Claim 8.2}}$

Claim 8.3. M^ω_θ *is a* (μ, θ)-d-*limit model*

Proof. $(\mathfrak{M}, \overline{a}, \mathfrak{N}, \mathcal{M}, \mathcal{N})^\omega$ is relatively full to $(M^n_i)_{n<\omega, i\in I_\omega}$ (by lemma 7.7). So, by proposition 7.6, $M^\omega_{i_\theta}$ is a (μ, θ)-d-limit witnessed by $\{M^\omega_i : i < \imath_\theta\}$ (notice that continuity of reduced towers guarantees that $M^\omega_{i_\theta} = \overline{\bigcup_{i<i_\theta} M^\omega_i}$). This finishes the proof of claim 8.3 $\square_{\text{Claim 8.3}}$

So, we have constructed a (μ, ω)-d-limit model over M which is also a (μ, θ)-d-limit model over M. $\square_{\text{Prop. 8.1}}$

Corollary 8.4. *If* M_i *is a* (μ, θ_i)-d*-limit over* M *(*$i \in \{1, 2\}$*), then* $M_1 \approx_M M_2$.

Proof. By proposition 8.1, they are isomorphic to a (μ, ω)-d-limit model over M. $\square_{\text{Cor. 8.4}}$

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